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CLASSIFICATION OF TRAVELING SALESMAN PROBLEM FORMULATIONS*

by

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ABSTRACT

Several "two-commodity" and "multi-commodity" flow formulations have recently been introduced for the traveling salesman problem. The purpose of this paper is to clarify the relations between these formulations and with other classical formulations.

Key words: Traveling Salesman Problem; Combinatorial Optimization; Linear Programming.

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RÉSUMÉ

Quelques formulations à deux et plusieurs ressources ont récemment été introduites pour le problème du voyageur de commerce. Le but de cet article est de clarifier les relations entre ces formulations et les formulations classiques.

Mots clés: problème du voyageur de commerce; optimisation combinatoire; programmation linéaire.

1. Classical formulations

The classical formulation of the traveling salesman problem consists in modeling it as an assignment problem (1–4) with integrality constraints (5) and subtour breaking constraints:

$$\text{Min} \quad \sum_i \sum_j c_{ij} x_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_i x_{ij} = 1 \qquad j = 1, \ldots, n \tag{2}$$

$$\sum_j x_{ij} = 1 \qquad i = 1, \ldots, n \tag{3}$$

$$0 \leq x_{ij} \leq 1 \qquad \text{for all } i, j \tag{4}$$

$$x_{ij} \text{ integer} \qquad \text{for all } i, j, \tag{5}$$

$$+ \text{ subtours breaking constraints.}$$

The coefficient c_{ij} represents the cost (or the distance) to travel from node i to node j. We state $c_{ii} = \infty$ for all i. The variable x_{ij} takes the value 1 if the route goes along arc (i, j) and 0 otherwise. The equations (2) and (3) form the assignment constraints and ensure that each node is visited only once.

Dantzig, Fulkerson and Johnson [2] use this formulation with the following subtours breaking constraints:

$$\sum_{i \in Q} \sum_{j \in Q} x_{ij} \leq |Q| - 1 \text{ for all } Q \subseteq \{1, 2, \ldots, n\} \text{ and } 2 \leq |Q| \leq n - 1. \tag{6}$$

This formulation (1–6) uses $O(n^2)$ binary variables and $O(2^n)$ constraints.

Miller, Tucker and Zemlin [11] present another formulation based on the assignment model but using a polynomial number of subtour breaking constraints by means of continuous variables. These constraints are:

$$u_i - u_j + n x_{ij} \leq n - 1 \text{ for all } j \neq 1 \text{ and } i \neq j. \tag{7}$$

This formulation requires $O(n^2)$ binary variables, $O(n)$ continuous variables and $O(n^2)$ constraints.

Finaly, Gavish and Graves [5] propose a formulation using $O(n^2)\, x_{ij}$ binary variables, $O(n^2)\, z_{ij}$ continuous variables and n^2+3n constraints. The variables z_{ij} describe the flow of a commodity from node one to each other node. The subtour breaking constraints are given by:

$$\sum_j z_{ij} - \sum_{j\neq 1} z_{ji} = 1 \quad i = 2,\ldots,n \tag{8}$$

$$z_{ij} \leq (n-1)x_{ij} \quad i = 2,\ldots,n \quad j = 1,\ldots,n \tag{9}$$

$$z_{ij} \geq 0 \quad \text{for all } i,j. \tag{10}$$

Let $v_{LP}(DFJ), v_{LP}(MTZ)$ and $v_{LP}(GG)$ denote the optimal value of the linear programming relaxations for the Dantzig, Fulkerson and Johnson, Miller, Tucker and Zemlin, and, Gavish and Graves formulations respectively. Wong [12] demonstrates that $v_{LP}(MTZ) \leq v_{LP}(GG) \leq v_{LP}(DFJ)$ and notes that the difference between $v_{LP}(GG)$ and $v_{LP}(DFJ)$ can be very large. Held and Karp [7] on their part experimentally observe that, for many problems, $v_{LP}(DFJ)$ is very close (within 0.5%) to the optimal solution of the traveling salesman problem.

2. Multi-commodity flow formulations

Wong [12] presents a formulation for the TSP using the flow of commodities $Y^k = (y_{ij}^k)$ and $Z^k = (z_{ij}^k)$:

(Wong) $\quad \text{Min} \sum_i \sum_j c_{ij} x_{ij}$

s.t. $\quad (2),(3),(4),(5)$

$$\sum_j (y_{ij}^k - y_{ji}^k) = \left\{ \begin{array}{ll} 1 & \text{if } i = 1 \\ -1 & \text{if } i = k \\ 0 & \text{if } i \neq 1 \text{ and } k \end{array} \right\} \quad k = 2,\ldots,n \tag{11}$$

$$\sum_j (z_{ij}^k - z_{ji}^k) = \left\{ \begin{array}{ll} -1 & \text{if } i = 1 \\ 1 & \text{if } i = k \\ 0 & \text{if } i \neq 1 \text{ and } k \end{array} \right\} \quad k = 2,\ldots,n \tag{12}$$

$$y_{ij}^k \leq x_{ij} \text{ and } z_{ij}^k \leq x_{ij} \qquad \text{for all } i,j,k \tag{13}$$

$$y_{ij}^k \geq 0, \qquad z_{ij}^k \geq 0 \qquad \text{for all } i,j,k. \tag{14}$$

Constraints (11) and (12) ensure that a unit of commodity Y^k travels from node 1 (source of commodity Y^k) to node k (the sink of commodity Y^k) while one unit of commodity Z^k travels from node k to node 1. This implies that the subgraph support of the flow is strongly connected i.e. that there is on this subgraph a path going from any node to any other node (passing by the source). This formulation requires $O(n^2)$ binary variables, $O(n^3)$ continuous variables and $2(n^3 + n^2 + n)$ constraints. Only the x_{ij} variables are subject to integrality since the y_{ij}^k and z_{ij}^k variables are necessarily integer whenever the x_{ij}'s are so, because the flow matrix for commodities Y^k and Z^k is totally unimodular (the x_{ij}'s correspond to the capacities on the arcs). Wong [12] demonstrates that

$$v_{LP}(Wong) = v_{LP}(DFJ) \tag{15}$$

and that thus it is possible to replace an exponential number of subtour breaking constraints by a polynomial number of constraints (at the expense of an additional number of variables).

Claus [1] relaxes Wong's formulation by eliminating the z_{ij}^k variables and the constraints in which they appear. Osbviously we have

$$v_{LP}(Claus) \leq v_{LP}(Wong) \tag{16}$$

because the Claus' formulation is a relaxation of Wong's. We demonstrate the equivalence of the two formulations with the following lemma:

Lemma 1:

$$v_{LP}(Wong) \leq v_{LP}(Claus)$$

Demonstration:

Let $X = (x_{ij})$ and $Y^k = (y_{ij}^k)$ $k = 2, \ldots, n$ be a solution satisfying Claus' formulation. Since $y_{ij}^k \leq x_{ij}$, define $z_{ij}^k = x_{ij} - y_{ij}^k \geq 0$. We then have $z_{ij}^k \leq x_{ij}$ for all i, j and

$$\sum_j (y_{ij}^k - y_{ji}^k) = \left\{ \begin{array}{ll} 1 & i = 1 \\ -1 & i = k \\ 0 & i \neq 1 \text{ and } k \end{array} \right\} k = 2, \ldots, n.$$

Replacing by the z_{ij}^k's, we obtain:

$$\sum_j \left((x_{ij} - z_{ij}^k) - (x_{ji} - z_{ji}^k) \right) = \left\{ \begin{array}{ll} 1 & i = 1 \\ -1 & i = k \\ 0 & i \neq 1 \text{ and } k \end{array} \right\} k = 2, \ldots, n.$$

As $\sum_j x_{ij} = \sum_j x_{ji} = 1$, then

$$\sum_j (z_{ij}^k - z_{ji}^k) = \left\{ \begin{array}{ll} -1 & i = 1 \\ 1 & i = k \\ 0 & i \neq 1 \text{ and } k \end{array} \right\} k = 2, \ldots, n.$$

We thus obtain a solution of Wong's formulation at the same cost. ∎

Langevin [8] introduces a formulation that replaces, for each arc (i, j), the two inequations (13) by only one more restrictive constraint

$$y_{ij}^k + z_{ij}^k \leq x_{ij} \text{ for all } i, j \text{ and } k. \tag{17}$$

Loulou [9] considers a formulation with even more restrictive constraints:

$$y_{ij}^k + z_{ij}^k = x_{ij} \text{ for all } i, j \text{ and } k. \tag{18}$$

Let us call Langevin and Loulou the LP relaxations corresponding respectively to (17) and (18). By using constraints more restrictive than that of Wong, the formulation with (17) or (18) apparently produces better bounds i.e.

$$v_{LP}(Wong) \leq v_{LP}(Langevin) \leq v_{LP}(Loulou). \tag{19}$$

We demonstrate by lemma 2 and the following theorem that these bounds are equivalent.

Lemma 2:

$$v_{LP}(Loulou) \leq v_{LP}(DFJ)$$

Demonstration:

Let $X = (x_{ij})$, be an optimal solution of (1–6), the LP relaxation of Dantzig, Fulkerson and Johnson. For each node k, the subtour breaking constraints ensure that a flow of at least one unit of X crosses towards k each cut between node 1 and node k. If we consider the x_{ij} variables as arc capacities, then the max. flow/min. cut theorem assures us that it is possible to define flows $Y^k = (y_{ij}^k)$, $k = 2, \ldots, n$ such that one unit of flow Y^k goes from node 1 to node k and hence (11) are satisfied:

$$\sum_j (y_{ij}^k - y_{ji}^k) = \left\{ \begin{array}{ll} 1 & i = 1 \\ -1 & i = k \\ 0 & i \neq 1 \text{ and } k \end{array} \right\} k = 2, \ldots, n$$

$$y_{ij}^k \leq x_{ij} \quad \text{for all } i, j, k.$$

The flow Y^k can be without loop: we then have $\sum_j y^k_{kj} = 0$ and $\sum_j y^k_{j1} = 0$. We can define flows $Z^k = X - Y^k$, $k = 2, \ldots, n$ and we show that one unit of flow Z^k goes from node k to node 1 (constraints (12)).

Let us have $z^k_{ij} = x_{ij} - y^k_{ij}$ for all i, j, k.

1) At node 1: we have $\sum_j x_{j1} = 0$ and $\sum_j y^k_{j1} = 0$
   then $\sum_j (x_{j1} - y^k_{j1}) = 1$ and then $\sum_j z^k_{j1} = 1$.
2) At node k: we have $\sum_j x_{kj} = 1$ and $\sum_j y^k_{kj} = 0$
   then $\sum_j (x_{kj} - y^k_{kj}) = 1$ and then $\sum_j z^k_{kj} = 1$.
3) Let $i(\neq 1$ and $\neq k)$ be any node.
   We have $\sum_j x_{ij} = \sum_j x_{ji}$ and $\sum_j y^k_{ij} = \sum_j y^k_{ji}$
   then $\sum_j (x_{ij} - y^k_{ij}) = \sum_j (x_{ji} - y^k_{ji})$ and then $\sum_j z^k_{ij} = \sum_j z^k_{ji}$.

Hence (X, Y, Z) is a solution of Loulou's relaxation at the same cost. ∎

Theorem 1:

$$v_{LP}(DFJ) = v_{LP}(Wong) = v_{LP}(Langevin) = v_{LP}(Loulou) = v_{LP}(Claus)$$

Demonstration:
If we apply successively (15), lemma 1, (16), (19) and lemma 2, we have
$v_{LP}(DFJ) = v_{LP}(Wong) \leq v_{LP}(Claus) \leq v_{LP}(Wong) \leq v_{LP}(Langevin) \leq v_{LP}(Loulou) \leq v_{LP}(DFJ)$, which demonstrates the theorem. ∎

Thus, all the LP relaxations of the multi-commodity flow formulations considered in this part give the same bound than the LP relaxation of the Dantzig, Fulkerson and Johnson's formulation. However, these LP formulations cannot be used to solve directly TSPs of substantial size because of the large number of constraints $[O(n^3)]$.

3. Two-commodity flow formulations

Finke, Claus and Gunn [3] propose a two-commodity flow formulation for the TSP. Consider that the traveling salesman leaves node 1 with $n - 1$ units of commodity Y and none of commodity Z. He distributes one unit of Y and picks up one unit of Z at each node so that the salesman travels at all times with a total of $n - 1$ units. The model writes as follows:

$$\text{Min} \sum_i \sum_j c_{ij}(y_{ij} + z_{ij})/n - 1 \tag{20}$$

$$\sum_j (y_{ij} - y_{ji}) = \begin{cases} n-1 & \text{for } i = 1 \\ 1 & \text{for } i \neq 1 \end{cases} \tag{21}$$

$$\sum_j (z_{ij} - z_{ji}) = \begin{cases} -(n-1) & \text{for } i = 1 \\ 1 & \text{for } i \neq 1 \end{cases} \tag{22}$$

$$\sum_j (y_{ij} + z_{ij}) = n - 1 \quad \text{for all } i \tag{23}$$

$$y_{ij} + z_{ij} \in \{0,\ n-1\} \quad \text{for all } i, j \tag{24}$$

$$y_{ij} \geq 0, \quad z_{ij} \geq 0 \quad \text{for all } i, j. \tag{25}$$

Constraints (21) and (22) define the flow conservation equations for each commodity. Constraints (23) and (24) ensure that there is exactly one arc supporting a combined flow of $n-1$ units out of each node. The supply and demand structure implies that there is a path from source 1 to each node j and another one from each node j to 1. Therefore, the graph support is strongly connected as in the case of the multi-commodity flow formulations.

The LP relaxation of the preceding formulation is obtained by replacing constraints (24) by $y_{ij} + z_{ij} \leq n-1$ for all i, j. These constraints are redundant because of (23) and can be eliminated from the LP relaxation. The corresponding LP, noted FCG, contains then $3n$ constraints whereas the integer formulation contains $O(n^2)$ additional constraints.

4. Relations between two-commodity and multi-commodity flow formulations

As first result, we can show that Finke, Claus and Gunn's formulation is obtained from Loulou's formulation. Constraints (21) and (22) are respectively obtained by summing on k the groups of equations (11) and (12) and by aggregating the commodities Y^k and Z^k such that:

$$y_{ij} = \sum_{k=2}^{n} y_{ij}^k \quad \text{and} \quad z_{ij} = \sum_{k=2}^{n} z_{ij}^k. \tag{26}$$

Furthermore, by summing on k equations (18) we obtain $y_{ij} + z_{ij} = (n-1)x_{ij}$, from which we obtain (24) since x_{ij} is binary. Constraints (23) are obtained by summing on j the preceding equation and by using the assignment equations.

Finke, Claus and Gunn [3] demonstrate that the bound obtained from their formulation is greater than or equal to the solution of the assignment problem and they give an example where their bound is strictly better than that of the corresponding assignment problem. Furthermore, Loulou [9] finds an example for which Finke, Claus and Gunn's bound is smaller than that of Wong. Then these bounds are not equivalent and we conclude that:

$$v_{LP}\ (\text{assignment}) \leq v_{LP}(FCG) \leq v_{LP}(Wong). \tag{27}$$

It is easy to show that the generalization of Finke, Claus and Gunn's formulation by Lucena [10] is also obtained from Loulou's formulation. In Lucena's formulation, the traveling salesman starts at node 1 and distributes d_i units of commodity Y and picks up d_i units of commodity Z at node i, $i = 2, \ldots, n$. Lucena's constraints are obtained by aggregating the constraints (11) and (12) with the ponderations d_k and by defining the flows by:

$$y_{ij} = \sum_{k=2}^{n} d_k y_{ij}^k \quad \text{and} \quad z_{ij} = \sum_{k=2}^{n} d_k z_{ij}^k. \tag{28}$$

5. Gavish and Graves vs Finke, Claus and Gunn.

We demonstrate by the following theorem the equivalence between the LP relaxations of the formulations of Gavish and Graves, and of Finke, Claus and Gunn. This result was unexpected since the GG formulation contains $O(n^2)$ constraints while the FCG formulation constains $O(n)$ constraints only.

Theorem 2:

$$v_{LP}(GG) = v_{LP}(FCG)$$

Demonstration:

A) Let (y_{ij}, z_{ij}) be a feasible solution of the LP relaxation of Finke, Claus and Gunn.
Let $x_{ij} = (y_{ij} + z_{ij})/(n-1)$.

1) $\sum_i \sum_j c_{ij}(y_{ij} + z_{ij})/(n-1) = \sum_i \sum_j c_{ij} x_{ij}$.

2) $\sum_j (y_{ij} + y_{ji}) = -1$ and $\sum_j (z_{ij} - z_{ji}) = 1 \Rightarrow \sum_j (y_{ij} + z_{ij}) - \sum_j (y_{ji} + z_{ji}) = 0$
$\Rightarrow \sum_j (x_{ij} - x_{ji}) = 0.$

3) $\sum_j (y_{ij} + z_{ij}) = (n-1) \Rightarrow \sum_j (n-1)x_{ij} = (n-1) \Rightarrow \sum_j x_{ij} = 1.$

4) $y_{ij} \geq 0$ and $y_{ij} = x_{ij}(n-1) - z_{ij} \Rightarrow x_{ij}(n-1) - z_{ij} \geq 0 \Rightarrow z_{ij} \leq x_{ij}(n-1)$.

Thus we obtain a solution of the LP relaxation of the formulation of Gavish and Graves at the same cost.

B) Let (x_{ij}, z_{ij}) be a feasible solution of the LP relaxation of Gavish and Graves.

Let $y_{ij} = x_{ij}(n-1) - z_{ij}$. We have $x_{ij} = (y_{ij} + z_{ij})/(n-1)$.

1) $\sum_i \sum_j c_{ij} x_{ij} = \sum_i \sum_j c_{ij}(y_{ij} + z_{ij})/(n-1)$.

2) $\sum_j (y_{ij} - y_{ji}) = \sum_j \Big(((n-1)x_{ij} - z_{ij}) - ((n-1)x_{ji} - z_{ji}) \Big)$

$= (n-1)\sum_j (x_{ij} - x_{ji}) - \sum_j (z_{ij} - z_{ji}) = 0 - \sum_j (z_{ij} - z_{ji}) = -1$ if $i \neq 1$.

We obtain the redundant equation $\sum_j (z_{ij} - z_{ji}) = -(n-1)$ by summing on all the other ones.

3) $\sum_j (y_{ij} + z_{ij}) = \sum_j x_{ij}(n-1) = (n-1)\sum_j x_{ij} = (n-1)$ for all i.

4) As $z_{ij} \leq (n-1)x_{ij}$ then $y_{ij} \geq 0$.

We then obtain a solution (y_{ij}, z_{ij}) of the LP relaxation of Finke, Claus and Gunn at the same cost. ∎

Finke, Claus and Gunn's formulation is so much more interesting that it is well suited to model TSP's with additional technological constraints. For example, Finke and Kusiak [4] use this model in the area of flexible manufacturing systems and Lucena [10] in the area of vehicle routing.

6. Conclusion.

We have demonstrated the equivalence of the LP relaxation of several multi-commodity flow formulations for the TSP. We have also demonstrated the equivalence of the LP relaxation of the multi-commodity flow formulations containing $O(n^2)$ constraints with that of the formulation of Dantzig, Fulkerson and Johnson [2] that contains $O(2^n)$ constraints. The equivalence of all these formulations enables to benefit from the experimental results obtained by all the authors. This allows using the formulation which is easiest to solve according to the type of approach since the corresponding bounds are all equivalent.

We showed that the formulation of Finke, Claus and Gunn [3] is obtained by aggregating the constraints and the variables of the multi-commodity flow formulation by Loulou [9]. Finally we have shown the equivalence of the LP relaxations of the formulation by Gavish and Graves [5] containing $O(n^2)$ constraints and that of Finke, Claus and Gunn [3] containing $3n$ constraints. The numerical experimentation of Gavish and Srikanth [6] on the lagrangian relaxation of the formulation of Gavish and Graves is giving us clues on what could be done with certain lagrangian relaxations of the formulation of Finke, Claus and Gunn.

All these results on the bounds of the LP relaxations are summarized in Figure 1.

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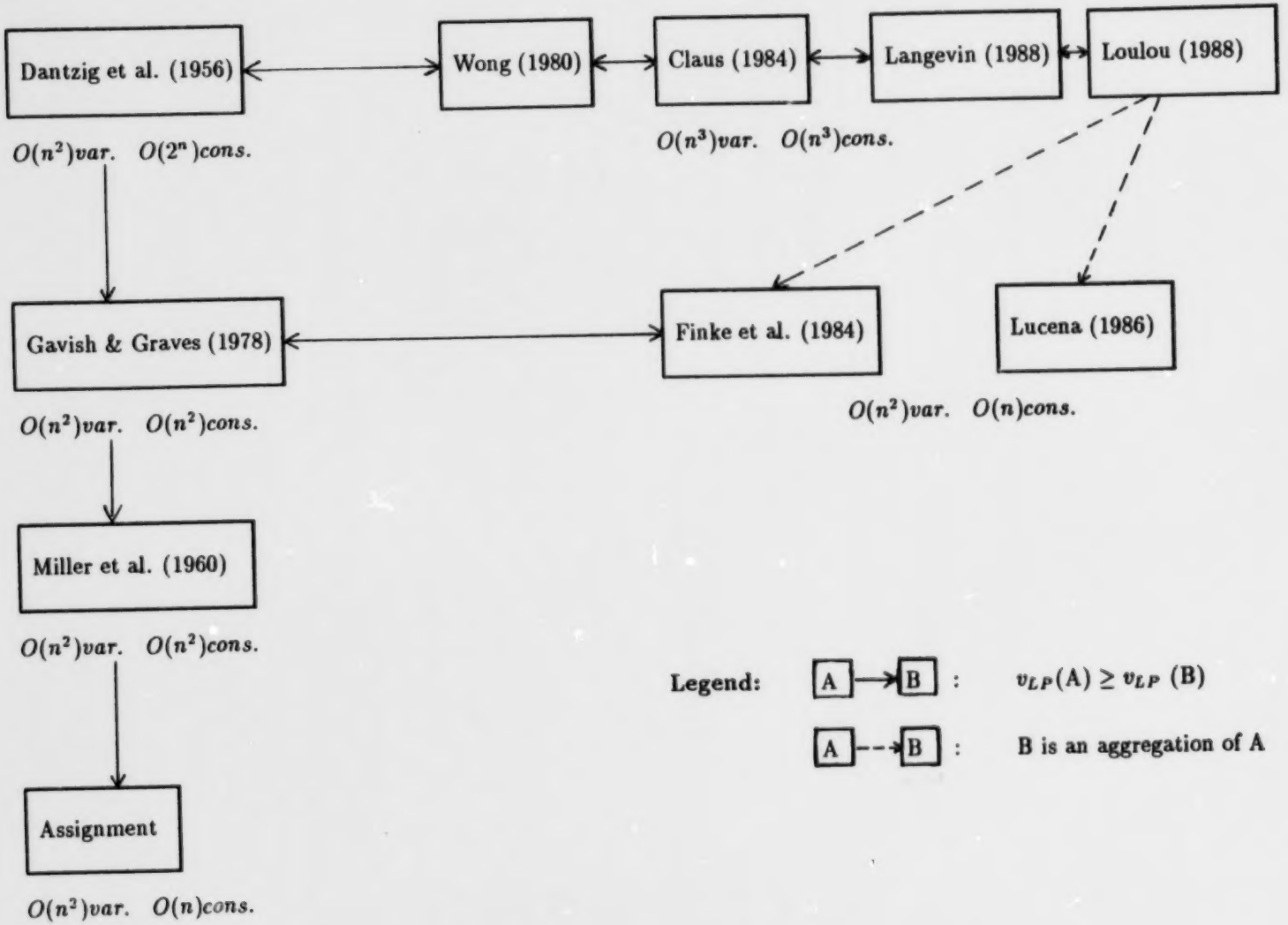

Figure 1: Classification of TSP formulation